On the intersection of additive perfect codes *

J. Rifà[‡], F. Solov'eva[†], M. Villanueva[‡]

November 13, 2006

Abstract

The intersection problem for additive (extended and non-extended) perfect codes, i.e. which are the possibilities for the number of codewords in the intersection of two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of the same length, is investigated. Lower and upper bounds for the intersection number are computed and, for any value between these bounds, codes which have this given intersection value are constructed.

For all these codes $\mathcal{C}_1$ and $\mathcal{C}_2$, the abelian group structure of the intersection codes $\mathcal{C}_1 \cap \mathcal{C}_2$ is characterized. The parameters of this abelian group structure corresponding to the intersection codes are computed and lower and upper bounds for these parameters are established. Finally, constructions of codes the intersection of which fits any parameters between these bounds are given.

Index Terms: intersection, additive codes, perfect codes, extended perfect codes.

1 Introduction and basic definitions

Let $\mathbb{F}^n$ be an n-dimensional vector space over the finite field $\mathbb{Z}_2$. The *Hamming distance* $d(v,s)$ between two vectors $v, s \in \mathbb{F}^n$ is the number of coordinates in which v and s differ.

A *binary code* C of length n is a subset of $\mathbb{F}^n$. The elements of a code are called *codewords*. The *minimum distance* d of a code C is the minimum

*This work has been partially supported by the Spanish MEC and the European FEDER Grant MTM2006-03250 and also by the UAB grant PNL2006-13.

[†]F. Solov'eva is with the Sobolev Institute of Mathematics, Novosibirsk, Russia. (email: sol@math.nsc.ru)

[‡]J. Rifà and M. Villanueva are with the Department of Information and Communications Engineering, Universitat Autònoma de Barcelona, 08193-Bellaterra, Spain. (email: {josep.rifa, merce.villanueva}@autonoma.edu)

value of $d(a, b)$, where $a, b \in C$ and $a \neq b$. The *error correcting capability* of a code C is the value $e = \lfloor \frac{d-1}{2} \rfloor$ and C is called an e-error correcting code. Two binary codes C_1 and C_2 of length n are *isomorphic* if there exists a coordinate permutation π such that $C_2 = \{\pi(c) \mid c \in C_1\}$. They are *equivalent* if there exist a vector $a \in \mathbb{F}^n$ and a coordinate permutation π such that $C_2 = \{a + \pi(c) \mid c \in C_1\}$.

A *binary perfect 1-error correcting code* (briefly in this paper, *binary perfect code*) C of length n is a subset of $\mathbb{F}^n$, with minimum distance $d = 3$, such that all the vectors in $\mathbb{F}^n$ are within distance one from a codeword. For any $t > 1$ there exists exactly one binary linear perfect code of length $2^t - 1$, up to equivalence, which is the well-known *Hamming code*. An *extended code* of the code C is a code resulting from adding an overall parity check digit to each codeword of C.

The intersection problem for binary perfect codes was proposed by Etzion and Vardy in [EV98]. They presented a complete solution of the intersection problem for binary Hamming codes: for each $t \geq 3$, there exist two Hamming codes H_1 and H_2 of length $n = 2^t - 1$ such that the number of codewords $\eta(H_1, H_2)$ in the intersection of these two codes is

$$\eta(H_1, H_2) = 2^{n-r} \quad \text{for} \quad r = t, t+1, \ldots, 2t. \tag{1}$$

They found the smallest intersection number for binary perfect codes of any admissible length consists of two codewords and investigated the intersection problem for binary perfect codes given by switchings of the binary Hamming codes. Last result was improved for binary perfect codes in [AHS05, AHS06] using switching approach. Bar-Yahalom and Etzion solved the intersection problem for q-ary cyclic codes in [BYE97]. The intersection problem for q-ary perfect codes is investigated in [SL06]. In [PV06], the intersection problem is also solved for Hadamard codes of length 2^t and of length $2^t s$ (s odd and $t \geq 6$), as long as there exists a Hadamard matrix of length $4s$.

The present paper is structured in the following way. This section contains the basic definitions about additive codes and their duals. These codes after a Gray map lead to the $\mathbb{Z}_4$-linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. We present some useful properties to manage those which are perfect codes and extended perfect codes. Section 2 is devoted to the generic class of additive codes establishing some results about the abelian group structure of the intersection and the intersection numbers for these codes. In Section 3 we settle the intersection problem for the additive extended perfect codes with $\alpha = 0$, i.e. for the quaternary linear perfect codes. We establish the lower and upper bounds for the intersection number and also for the parameters of the abelian group structure of the intersection of these codes. Moreover, we prove the

existence of codes with all the allowed parameters between these bounds. Finally, Section 4 reaches the same results than Section 3 but now for additive extended perfect codes with $\alpha \neq 0$.

1.1 Additive codes

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ be the ring of integers modulo 2 and modulo 4, respectively. Let $\mathbb{F}^n$ be the set of all binary vectors of length n and let $\mathbb{Z}_4^n$ be the set of all quaternary vectors of length n. As we said before, any non-empty subset C of $\mathbb{F}^n$ is a binary code and a subgroup of $\mathbb{F}^n$ is called a *binary linear code* or a $\mathbb{Z}_2$*-linear code*. Equivalently, any non-empty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a subgroup of $\mathbb{Z}_4^n$ is called a *quaternary linear code*.

Let $\mathcal{C}$ be a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and let $C = \Phi(\mathcal{C})$, where $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, $n = \alpha + 2\beta$, is given by $\Phi(x, y) = (x, \phi(y))$ for any x from $\mathbb{Z}_2^\alpha$ and any y from $\mathbb{Z}_4^\beta$, where $\phi : \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^{2\beta}$ is the usual Gray map, that is, $\phi(y_1, \dots, y_\beta) = (\varphi(y_1), \dots, \varphi(y_\beta))$, and $\varphi(0) = (0, 0), \varphi(1) = (0, 1), \varphi(2) = (1, 1)$, $\varphi(3) = (1, 0)$.

Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is also isomorphic to an abelian structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, we have that $|\mathcal{C}| = 2^\gamma 4^\delta$ and the number of order two codewords in $\mathcal{C}$ is $2^{\gamma+\delta}$. We call such code $\mathcal{C}$ an *additive code of type* $(\alpha, \beta; \gamma, \delta)$ and the binary image $C = \Phi(\mathcal{C})$ a $\mathbb{Z}_2\mathbb{Z}_4$*-linear code of type* $(\alpha, \beta; \gamma, \delta)$. In the specific case $\alpha = 0$ we see that $\mathcal{C}$ is a quaternary linear code and the code C is called a $\mathbb{Z}_4$*-linear code*. Note that the length of the binary code $C = \Phi(\mathcal{C})$ is $n = \alpha + 2\beta$.

Moreover, although $\mathcal{C}$ could not have a basis, it is interesting and adequate to define a generator matrix for $\mathcal{C}$ as:

$$\mathcal{G} = \left(\begin{array}{c|c} B_2 & Q_2 \\ \hline B_1 & Q_1 \end{array} \right),$$

where B_2 is a $\gamma \times \alpha$ matrix; Q_2 is a $\gamma \times \beta$ matrix; B_1 is a $\delta \times \alpha$ matrix and Q_1 is a $\delta \times \beta$ matrix. Matrices B_1, B_2 are binary and Q_1, Q_2 are quaternary, but the entries in Q_2 are only zeroes or twos. In what follows we denote the additive span of the union of two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ by $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$.

Notice that in some cases we will need to refer the type of an additive code as $(\alpha, \beta; \gamma, \delta; \kappa)$, where κ comes from the following consideration. Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the code $\mathcal{C}$ restricted to the X (respectively Y) coordinates. Let $\mathcal{D}$ be the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $\mathcal{D}_X$, which is a binary linear code. For the case $\alpha = 0$, we will write $\kappa = 0$.

Two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ both of the same length are said to be *equivalent*, if one can be obtained from the other by permuting the coordinates and changing the signs of certain coordinates. Additive codes which differ only by a permutation of coordinates are said to be *isomorphic*.

1.2 Duality of additive codes

We will use the following definition (see [RP97]) of the inner product in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$:

$$\langle u, v\rangle = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4, \tag{2}$$

where $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Note that when $\alpha = 0$ the inner product is the usual one for $\mathbb{Z}_4$-vectors (i.e. vectors over $\mathbb{Z}_4$) and when $\beta = 0$ it is twice the usual one for $\mathbb{Z}_2$-vectors.

We can also write

$$\langle u, v\rangle = u \cdot J_n \cdot v^T, \tag{3}$$

where $J_n = \left(\begin{array}{c|c} 2I_\alpha & 0 \\ \hline 0 & I_\beta \end{array}\right)$ is a diagonal quaternary matrix.

The *additive dual code* of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is defined in the standard way

$$\mathcal{C}^\perp = \{u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid \langle u, v\rangle = 0 \text{ for all } v \in \mathcal{C}\}.$$

The corresponding binary code $\Phi(\mathcal{C}^\perp)$ is denoted by $C_\perp$ and called the $\mathbb{Z}_2\mathbb{Z}_4$-*dual code* of C. In the case $\alpha = 0$, $\mathcal{C}^\perp$ is also called the *quaternary dual code* of $\mathcal{C}$ and $C_\perp$ the $\mathbb{Z}_4$-*dual code* of C.

The additive dual code $\mathcal{C}^\perp$ is also an additive code, that is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Its weight enumerator polynomial is related to the weight enumerator polynomial of $\mathcal{C}$ by McWilliams identity (see [Del73]). Notice that C and $C_\perp$ are not dual in the binary linear sense but the weight enumerator polynomial of $C_\perp$ is the McWilliams transform of the weight enumerator polynomial of C. So, we have (see [RP97])

$$|\mathcal{C}||\mathcal{C}^\perp| = 2^{\alpha+2\beta}. \tag{4}$$

It is known (see [BF+06]) that the additive dual code of an additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$, denoted by $\mathcal{C}^\perp$, is of type $(\alpha, \beta; \gamma', \delta'; \kappa')$, where

$$\begin{aligned} \gamma' &= \alpha + \gamma - 2\kappa, \\ \delta' &= \beta - \gamma - \delta + \kappa, \\ \kappa' &= \alpha - \kappa. \end{aligned} \tag{5}$$

Notice that given a quaternary linear code $\mathcal{C}$ of type $(0, \beta; \gamma, \delta)$, the additive dual code $\mathcal{C}^\perp$ is of type (see [HK+94])

$$(0, \beta; \gamma, \beta - \gamma - \delta). \tag{6}$$

Because the type of any additive code $\mathcal{C}$ is uniquely defined by the type of its dual code, for the sake of brevity we further will use the notation an *additive code* $\mathcal{C}$ *of dual type* $(\alpha, \beta; \gamma', \delta')$ which means that its additive dual code $\mathcal{C}^\perp$ is an additive code of type $(\alpha, \beta; \gamma', \delta')$.

1.3 Additive extended perfect codes

In all this paper we will take the permission to call *additive perfect codes* to additive codes such that, after the Gray map, give perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. Also, we will call *additive extended perfect codes* to additive codes such that, after the Gray map, give a code with the parameters of an extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code.

Apart from the linear binary case, so the case when $\beta = 0$, there are two different kinds of additive extended perfect codes, those with $\alpha \neq 0$ and those with $\alpha = 0$. We will distinguish between these two cases because the constructions are different.

Theorem 1 *[BR99] For each natural number r, such that $2 \leq r \leq t \leq 2r$, there exists a unique (up to isomorphism) perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of binary length $n = 2^t - 1 \geq 15$, such that the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C is of type $(\alpha, \beta; \gamma, \delta)$, where $\alpha = 2^r - 1, \beta = 2^{t-1} - 2^{r-1}, \gamma = 2r - t$ and $\delta = t - r$.*

After this theorem we can write the following table (see [BR99]):

t	r	(α, β)
2	2	$(3, 0)$
3	2,3	$(3, 2)$, $(7, 0)$
4	2,3,4	$(3, 6)$, $(7, 4)$, $(15, 0)$
5	3,4,5	$(7, 12)$, $(15, 8)$, $(31, 0)$
6	3,4,5,6	$(7, 28)$, $(15, 24)$, $(31, 16)$, $(63, 0)$
...	...	...

Note that for length 7 the two codes are isomorphic and for length greater than 7 all the codes in the table are non-isomorphic and unique. The number of non-isomorphic perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of length $n = 2^t - 1$ is $\left\lfloor \dfrac{t+2}{2} \right\rfloor$ for all $t > 3$, and it is 1 for $t = 2$ and $t = 3$.

For any r and $t \geq 4$ such that $2 \leq r \leq t \leq 2r$, there is exactly one extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C' with $\alpha = 2^r$ and $\beta = 2^{t-1} - 2^{r-1}$, up to

isomorphism. The corresponding additive codes of these perfect (extended perfect) $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are additive perfect (extended perfect) codes with $\alpha \neq 0$.

Notice that for these extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (or the corresponding additive extended perfect codes), we do not need to specify the parameter κ because $\kappa = \gamma$ (see [BF+06]), so we just talk about additive perfect codes of type $(\alpha, \beta; \gamma, \delta)$. Also notice that in this case given an additive perfect code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta)$, the additive dual code $\mathcal{C}^\perp$ is of type $(\alpha, \beta; \alpha - \gamma, \beta - \delta)$.

Example 1 *There are three non-isomorphic perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of length 15. They exist for $(r = 2, t = 4), (r = 3, t = 4)$ and $(r = 4, t = 4)$. For the case $(r = 3, t = 4)$ we have the perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of dual type $(7, 4; 2, 1)$ with the following parity-check matrix:*

$$\left(\begin{array}{ccccccc|cccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

The extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C' for this case is of dual type $(8, 4; 3, 1)$ with the following parity-check matrix:

$$\left(\begin{array}{cccccccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 2 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

Theorem 2 *[Kr01] For each $\delta \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$ there exists a unique (up to isomorphism) extended perfect $\mathbb{Z}_4$-linear code C' of binary length $n + 1 = 2^t \geq 16$, such that the $\mathbb{Z}_4$-dual code of C' is of type $(0, \beta; \gamma, \delta)$, where $\beta = 2^{t-1}$ and $\gamma = t + 1 - 2\delta$.*

Example 2 *In the case of length $n+1 = 32$, there are three non-isomorphic extended perfect $\mathbb{Z}_4$-linear codes, since we have three possible parameters: $\delta = 1$, $\delta = 2$ and $\delta = 3$. The following matrix is the parity-check matrix of the code C' for $\delta = 2$ (also notice that $\beta = 16$ and $\gamma = 2$):*

$$\left(\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{array}\right).$$

The corresponding additive codes of the extended perfect $\mathbb{Z}_4$-linear codes are additive extended perfect codes with $\alpha = 0$. It was established in [BF03] that if the code C' is an extended perfect $\mathbb{Z}_4$-linear code of binary length $n + 1 = 2^t \geq 16$, then the punctured code is not a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code, up to the extended Hamming code of length 16.

The study of additive extended perfect codes is absolutely different if they come from the extended code of a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code as in Theorem 1 or from extended perfect $\mathbb{Z}_4$-linear codes as in Theorem 2. Note that, in the first case, the vector with binary ones in the binary part and quaternary twos in the quaternary part is always in both codes, the additive extended perfect code and its additive dual code. However, in the second case, the quaternary all-ones vector is always in these two codes, the additive extended perfect code and its additive dual code.

2 Intersection of additive codes

In this section we consider the intersection problem for generic additive codes, with the same length and the same parameters α and β.

We will use the same starting point as in [EV98] and [BYE97]. Let $\mathcal{C}_1, \mathcal{C}_2$ be two additive codes. From the well-known second theorem of isomorphism for groups, we can write:

$$\langle \mathcal{C}_1, \mathcal{C}_2 \rangle / \mathcal{C}_1 \cong \mathcal{C}_2 / (\mathcal{C}_1 \cap \mathcal{C}_2). \tag{7}$$

Lemma 1 *Let $\mathcal{C}_1, \mathcal{C}_2$ be two additive codes, then $\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle = (\mathcal{C}_1 \cap \mathcal{C}_2)^\perp$.*

Proof: It is straightforward to see that $\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle \subset (\mathcal{C}_1 \cap \mathcal{C}_2)^\perp$. Moreover, the code $\mathcal{C}_1 \cap \mathcal{C}_2$ is the largest subgroup of both codes $\mathcal{C}_1$ and $\mathcal{C}_2$ and, hence, the code $(\mathcal{C}_1 \cap \mathcal{C}_2)^\perp$ is the lowest group containing both codes $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$. Any other group containing both codes $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ also contains $(\mathcal{C}_1 \cap \mathcal{C}_2)^\perp$. This is the case for the group $\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle$. $\square$

Like in the binary case (see [EV98]), the above lemma allows us the following interpretation. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be parity-check matrices of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then, $\begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$ is a parity-check matrix for the intersection code $\mathcal{C}_1 \cap \mathcal{C}_2$. For the sake of brevity as in [EV98] we further denote this last matrix by $\mathcal{H}_1 \parallel \mathcal{H}_2$.

We begin considering the case $\alpha = 0$, and after that, we investigate the additive case, which is more complicated.

Proposition 1 *For any two quaternary linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $(0, \beta; \gamma_1, \delta_1)$ and $(0, \beta; \gamma_2, \delta_2)$ respectively, the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is a quaternary linear code of type $(0, \beta; \gamma, \delta)$, where*

$$\delta \in \{max(\delta_1, \delta_2), \ldots, min(\delta_1 + \delta_2, \beta)\} \quad \textit{and} \tag{8}$$

$$max\big(\delta, max(\gamma_1 + \delta_1, \gamma_2 + \delta_2)\big) \leq \gamma + \delta \leq min(\gamma_1 + \gamma_2 + \delta_1 + \delta_2, \beta). \tag{9}$$

Proof: Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be generator matrices of arbitrary quaternary linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length β and types $(0, \beta; \gamma_1, \delta_1)$ and $(0, \beta; \gamma_2, \delta_2)$. Consider the matrix $\mathcal{G}_1 \parallel \mathcal{G}_2$, which is a generator matrix of the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ of type $(0, \beta; \gamma, \delta)$. After some additive transformations we get from the matrix $\mathcal{G}_1 \parallel \mathcal{G}_2$ a quaternary matrix, denoted by $\mathcal{G}$, with γ rows of order two and δ rows of order four. It is not difficult to see that

$$max(\delta_1, \delta_2) \leq \delta.$$

Each codeword of order two in the matrix $\mathcal{G}$ is a linear combination of rows of order two in the matrices $\mathcal{G}_1$ and $\mathcal{G}_2$, so the total number $\gamma + \delta$ of rows of order two in the matrix $\mathcal{G}$ is not more than $\gamma_1 + \delta_1 + \gamma_2 + \delta_2$, i.e. $\gamma + \delta \leq \gamma_1 + \delta_1 + \gamma_2 + \delta_2$. Moreover, $\gamma + \delta$ is not greater than the number β of coordinates.

On the other hand, $\delta \leq \delta_1 + \delta_2$ because in some cases there are rows of order two of the matrix $\mathcal{G}$ that can be obtained by additive combinations of rows of order four in the matrix $\mathcal{G}_1 \parallel \mathcal{G}_2$. Also, it must be $\delta \leq \beta$.

Let us fix s from the set $\{0, 1, \ldots, min(\delta_1, \delta_2)\}$ and suppose that s rows of order four among all $\delta_1 + \delta_2$ rows of order four in the matrix $\mathcal{G}_1 \parallel \mathcal{G}_2$ can be obtained by linear combinations of rows of the matrix $\mathcal{G}$. Then, we immediately get $\delta = \delta_1 + \delta_2 - s$ independent rows of order four in the matrix $\mathcal{G}$. Taking into account this, we get not less than $max(\gamma_1 + \delta_1, \gamma_2 + \delta_2) - \delta$ rows of order two in the matrix $\mathcal{G}$. Using the fact that in some cases the number $max(\gamma_1 + \delta_1, \gamma_2 + \delta_2) - \delta$ can be less than zero, we get the lower bound in (9). $\square$

Now we can generalize this last proposition to cover all the additive codes.

Proposition 2 *For any two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $(\alpha, \beta; \gamma_1, \delta_1; \kappa_1)$ and $(\alpha, \beta; \gamma_2, \delta_2; \kappa_2)$ respectively, the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is an additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, where*

$$\delta \in \{max(\delta_1, \delta_2), \ldots, min(\delta_1 + \delta_2, \beta)\}, \tag{10}$$

$$max\big(\delta, max(\kappa_1 + \delta_1, \kappa_2 + \delta_2)\big) \leq \kappa + \delta \leq min(\kappa_1 + \kappa_2 + \delta_1 + \delta_2, \alpha + \beta) \tag{11}$$

$$\textit{and} \quad \kappa + \delta \leq \gamma + \delta \leq min(\gamma_1 + \gamma_2 + \delta_1 + \delta_2, \alpha + \beta). \tag{12}$$

Proof: We use the same argumentation than in Proposition 1. Taking into account that always $\kappa \leq \gamma$ we can easily get the equations (10) and (12).

Now, to obtain (11) we construct two auxiliaries codes $\mathcal{C}_1'$ and $\mathcal{C}_2'$ from the given ones. In the generator matrices of the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ erase, respectively, the $\gamma_1 - \kappa_1$ and $\gamma_2 - \kappa_2$ rows which are dependent on the rest of rows when we restrict them to the binary part. These new codes $\mathcal{C}_1'$ and $\mathcal{C}_2'$ have the same lengths as the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ and are of type $(\alpha, \beta; \kappa_1, \delta_1; \kappa_1)$ and $(\alpha, \beta; \kappa_2, \delta_2; \kappa_2)$, respectively.

We have $\langle \mathcal{C}_1', \mathcal{C}_2' \rangle \subseteq \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ and using the same argumentation than in Proposition 1 we get the equation (11), since the parameter γ in $\langle \mathcal{C}_1', \mathcal{C}_2' \rangle$ coincides with the parameter κ in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. $\Box$

We finish this section describing the intersection numbers for additive codes. This means to describe the intersection code just talking about the cardinality and not about the abelian group structure. Next lemma allows us to compute the size for the additive dual code.

Lemma 2 *For any two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $(\alpha, \beta; \gamma_1, \delta_1; \kappa_1)$ and $(\alpha, \beta; \gamma_2, \delta_2; \kappa_2)$ respectively, the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is an additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, where the size $|\langle \mathcal{C}_1, \mathcal{C}_2 \rangle| = 2^\gamma 4^\delta$ satisfies the conditions:*

$$max(\kappa_1 + \delta_1, \kappa_2 + \delta_2) + max(\delta_1, \delta_2) \leq \gamma + 2\delta \leq \mu, \tag{13}$$

where $\mu = min(\gamma_1 + \gamma_2 + 2(\delta_1 + \delta_2), \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \beta, \delta_1 + \delta_2 + \alpha + \beta, \alpha + 2\beta)$.

Proof: By Proposition 2 we have

$$\gamma + \delta \leq min(\gamma_1 + \gamma_2 + \delta_1 + \delta_2, \alpha + \beta) \quad \text{and}$$

$$\delta \leq min(\delta_1 + \delta_2, \beta).$$

Therefore

$$\gamma + 2\delta \leq \mu,$$

where $\mu = min(\gamma_1 + \gamma_2 + 2(\delta_1 + \delta_2), \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \beta, \delta_1 + \delta_2 + \alpha + \beta, \alpha + 2\beta)$.

To get the lower bound for $\gamma + 2\delta$ we consider the lower bound in (11) and distinguish two cases.

Case 1: Let $max(\kappa_1 + \delta_1, \kappa_2 + \delta_2) \leq \delta$. From (11) we get $0 \leq \kappa \leq \gamma$. Last two inequalities and the lower bound $max(\delta_1, \delta_2) \leq \delta$ give us

$$max(\kappa_1 + \delta_1, \kappa_2 + \delta_2) + max(\delta_1, \delta_2) \leq \gamma + 2\delta.$$

Case 2: For the case $max(\kappa_1 + \delta_1, \kappa_2 + \delta_2) \geq \delta$ we have from (11) and (12) the following inequalities: $max(\kappa_1 + \delta_1, \kappa_2 + \delta_2) \leq \kappa + \delta \leq \gamma + \delta$ and, using the bound $max(\delta_1, \delta_2) \leq \delta$, we immediately get the lower bound in (13). $\Box$

Theorem 3 *For any two additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \gamma_1, \delta_1; \kappa_1)$ and $(\alpha, \beta; \gamma_2, \delta_2; \kappa_2)$ respectively, it is true that*

$$2^{\alpha+2\beta-\mu} \leq \eta(\mathcal{C}_1, \mathcal{C}_2) \leq 2^{\alpha+2\beta-max(\kappa_1+\delta_1,\kappa_2+\delta_2)-max(\delta_1,\delta_2)},$$

where $\mu = min(\gamma_1+\gamma_2+2(\delta_1+\delta_2), \gamma_1+\gamma_2+\delta_1+\delta_2+\beta, \delta_1+\delta_2+\alpha+\beta, \alpha+2\beta)$.

Proof: Using Lemma 2 one can easily get

$$2^{max(\kappa_1+\delta_1,\kappa_2+\delta_2)+max(\delta_1,\delta_2)} \leq |\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle| \leq 2^{\mu}. \tag{14}$$

By Lemma 1 and equation (4) we have

$$|\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle| = |(\mathcal{C}_1 \cap \mathcal{C}_2)^\perp| = 2^{\alpha+2\beta}/|(\mathcal{C}_1 \cap \mathcal{C}_2)|.$$

Hence, $\eta(\mathcal{C}_1, \mathcal{C}_2) = 2^{\alpha+2\beta}/|\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle|$ and the statement follows. □

3 Intersection of quaternary linear perfect codes

In this section we consider additive extended perfect codes such that $\alpha = 0$, which will also be called *quaternary linear perfect codes*. We investigate the intersection problem for such codes and the abelian group structure for their intersection codes. In Subsection 3.1 we consider the intersection problem and in Subsection 3.2 the abelian group structure of these intersection codes.

3.1 Intersection problem of quaternary linear perfect codes

All statements presented in the previous section are valid for the quaternary linear perfect codes with some modifications. We are going to omit some proofs indicating some specific properties of these codes. So, taking into account that the quaternary all-ones vector always belongs to the quaternary dual of any quaternary linear perfect code, we immediately get from Proposition 1:

Proposition 3 *For any two quaternary linear perfect codes such that their quaternary dual codes are $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $(0, \beta; \gamma_1, \delta_1)$ and $(0, \beta; \gamma_2, \delta_2)$ respectively, with $\beta = 2^{t-1}$ and $t \geq 4$, the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is a quaternary linear code of type $(0, \beta; \gamma, \delta)$, where*

$$\delta \in \{max(\delta_1, \delta_2), \ldots, \delta_1 + \delta_2 - 1\} \quad and \tag{15}$$

$$max\big(\delta, max(\gamma_1 + \delta_1, \gamma_2 + \delta_2)\big) \leq \gamma + \delta \leq \gamma_1 + \gamma_2 + \delta_1 + \delta_2 - 1. \tag{16}$$

Proof: The quaternary linear perfect codes satisfy (see Theorem 2) $\beta = 2^{t-1}$, $\gamma_1 + 2\delta_1 = \gamma_2 + 2\delta_2 = t+1$ and $\delta_1, \delta_2 \in \{1, \dots, \lfloor (t+1)/2 \rfloor\}$.

The quaternary all ones vector is always in the quaternary dual, which means that the upper bound for δ is not $min(\delta_1+\delta_2, \beta)$ but $min(\delta_1+\delta_2-1, \beta)$. Since $t \geq 4$ using Theorem 2 we have $\delta_1 + \delta_2 - 1 \leq t \leq \beta$, so we can write $\delta_1 + \delta_2 - 1$ as the upper bound for δ in equation (8).

Moreover taking into account that $\gamma_1 + 2\delta_1 = \gamma_2 + 2\delta_2 = t+1$ and $\delta_1 \geq 1, \delta_2 \geq 1$ we get $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 - 1 = 2(t+1) - \delta_1 - \delta_2 - 1 \leq 2t - 1$. Again, since $t \geq 4$ we have $2t - 1 < \beta$, so we can write $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 - 1$ as the upper bound for $\gamma + \delta$ in equation (9). This proves the statement. □

Next two theorems give us the solution of the intersection problem for quaternary linear perfect codes. First, we present the lower and the upper bounds. Then, we show that there exist such codes for any possible intersection number between these bounds.

Theorem 4 *For any $t \geq 3$ and any quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length $\beta = 2^{t-1}$, it is true that*

$$2^{2\beta - 2t} \leq \eta(\mathcal{C}_1, \mathcal{C}_2) \leq 2^{2\beta - t - 1}.$$

Proof: We know that for the quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ the quaternary all-ones vector is in their quaternary dual codes, so the statement in Theorem 3 is

$$2^{2\beta - \mu} \leq \eta(\mathcal{C}_1, \mathcal{C}_2) \leq 2^{2\beta - max(\gamma_1 + \delta_1, \gamma_2 + \delta_2) - max(\delta_1, \delta_2)},$$

where using Proposition 3 we immediately get $\mu = 2t$. Also, by Theorem 2, we know that $\beta = 2^{t-1}$ and $\gamma_1 + 2\delta_1 = \gamma_2 + 2\delta_2 = t+1$, so for $t \geq 3$ we have:

$$max(\gamma_1 + \delta_1, \gamma_2 + \delta_2) + max(\delta_1, \delta_2) = t + 1 + |\delta_2 - \delta_1|.$$

Therefore

$$2^{2\beta - max(\gamma_1 + \delta_1, \gamma_2 + \delta_2) - max(\delta_1, \delta_2)} = 2^{2\beta - (t+1) - |\delta_2 - \delta_1|} \leq 2^{2\beta - t - 1},$$

so the statement follows. □

The lower bound given by Theorem 4 is an even power of two. So, comparing with the intersection problem for binary Hamming codes, see (1) above, it is impossible to get two extended perfect $\mathbb{Z}_4$-linear codes of length $2\beta = 2^t$ $(t \geq 3)$ with intersection number $2^{2\beta - 2t - 1}$.

Theorem 5 *For any $t \geq 3$ there exist two quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length $\beta = 2^{t-1}$, such that $\eta(\mathcal{C}_1, \mathcal{C}_2) = 2^{2\beta - l}$, where l is any value from $t+1$ to $2t$.*

Proof: By Theorem 4, the minimum and maximum intersection numbers for quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length $\beta = 2^{t-1}$ are $2^{2\beta-2t}$ and $2^{2\beta-t-1}$, respectively.

For $t = 3$, we have to find quaternary linear perfect codes of length $\beta = 4$ with intersection numbers 16, 8 and 4. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the quaternary linear perfect codes with the parity-check matrices

$$\left(\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cccc} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 1 & 1 & 1 & 1 \end{array}\right),$$

respectively. Then, it is easy to see that $\eta(\mathcal{C}_1, \mathcal{C}_1) = 16$, $\eta(\mathcal{C}_1, \mathcal{C}_2) = 8$ and $\eta(\mathcal{C}_1, \pi(\mathcal{C}_1)) = 4$, where $\pi = (1, 2)$.

For $t \geq 4$, we consider the parity-check matrix of the quaternary linear perfect code of length β with $\gamma = t-1$ and $\delta = 1$. This parity-check matrix can be represented as the quaternary matrix

$$\mathcal{H} = \left(\begin{array}{c} 2H \\ \hline 1 \dots 1 \end{array}\right),$$

where H is the matrix with the first column the all-zeroes vector of length $t-1$ and the rest of the columns from the parity-check matrix of a binary Hamming code of length $\beta - 1$. By the classification of intersection numbers of binary Hamming codes given by Etzion and Vardy in [EV98], the rank of the matrix $H \parallel \pi(H)$ can vary from $t-1$ till $2(t-1)$ for different permutations π of length β. Then, there exist quaternary linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length β with parity-check matrices $\mathcal{H}$ and $\pi(\mathcal{H})$ respectively, such that the code $\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle$ with generator matrix $\mathcal{H} \parallel \pi(\mathcal{H})$ is of type $(0, \beta; r, 1)$ for all $r \in \{t-1, \dots, 2t-2\}$. Since $|\mathcal{C}_1 \cap \mathcal{C}_2| \cdot |\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle| = 2^{2\beta}$ and $|\langle \mathcal{C}_1^\perp, \mathcal{C}_2^\perp \rangle| = 2^r 4$, we have the intersection numbers $\eta(\mathcal{C}_1, \mathcal{C}_2) = 2^{2\beta-(r+2)} = 2^{2\beta-l}$, where $l = r+2$ is any value from $t+1$ to $2t$. $\square$

3.2 The abelian group structure for the intersection of quaternary linear perfect codes

To investigate the abelian group structure for the intersection of quaternary linear perfect codes it will be helpful the following statement, which we can get immediately from Proposition 3.

Theorem 6 *For any two quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0,\beta;\gamma_1,\delta_1)$ and $(0,\beta;\gamma_2,\delta_2)$ respectively, with $\beta = 2^{t-1}$ and $t \geq 4$, the intersection code $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(0,\beta;\gamma,\delta)$, where γ and δ satisfy the bounds given by Proposition 3.*

From this theorem, it is easy to compute the type of the intersection code $\mathcal{C}_1 \cap \mathcal{C}_2$ using the fact that if a quaternary linear code has parameters $(0,\beta;\gamma,\delta)$, then its quaternary dual code has parameters $(0,\beta;\gamma,\beta-\gamma-\delta)$.

Next we will show that there exist quaternary linear perfect codes of length $\beta = 2^{t-1}$ for any $t \geq 4$, with intersections of type $(0,\beta;\gamma,\delta)$ for all possible γ and δ between the bounds given by Theorem 6. In Example 3, for quaternary linear perfect codes of length $\beta = 4$ $(t = 3)$, we show which intersections codes with parameters between these bounds are possible.

Example 3 *For $\beta = 4$ $(t = 3)$ there are two isomorphic quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ given by $\delta = 1$ and $\delta = 2$, so of dual types $(0,4;2,1)$ and $(0,4;0,2)$, respectively. We can take*

$$\mathcal{H}_1 = \left(\begin{array}{cccc} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 1 & 1 & 1 & 1 \end{array}\right) \quad \textit{and} \quad \mathcal{H}_2 = \left(\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

as parity-check matrices of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

By an exhaustive search, the intersection code $\mathcal{C}_1 \cap \pi(\mathcal{C}_1)$ with parity-check matrix $\mathcal{H}_1 \parallel \pi(\mathcal{H}_1)$ is of dual type $(0,4;2,1)$ for any permutation π. On the other hand, the intersection code $\mathcal{C}_2 \cap \pi(\mathcal{C}_2)$ is either of dual type $(0,4;0,2)$ or $(0,4;0,3)$. For example, taking $\pi = Id$ and $\pi = (1,2)$, we find that

$$\left(\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right) \quad \textit{and} \quad \left(\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{array}\right)$$

are parity-check matrices of these two intersection codes. Finally, the intersection code $\mathcal{C}_1 \cap \pi(\mathcal{C}_2)$ is always of dual type $(0,4;1,2)$ for any permutation π.

Lemma 3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be quaternary linear perfect codes of dual type $(0,\beta;\gamma_1,\delta_1)$ and $(0,\beta;\gamma_2,\delta_2)$ respectively, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(0,\beta;i,j)$, $\beta \geq 4$. Then, there exist two quaternary linear perfect codes of dual type $(0,2\beta;\gamma_1+1,\delta_1)$ and $(0,2\beta;\gamma_2+1,\delta_2)$ with intersection codes of dual type $(0,2\beta;i',j)$ for $i' \in \{i+1,i+2\}$.*

Proof: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be parity-check matrices of the quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. The matrices

$$\begin{pmatrix} 0\ldots 0 & 2\ldots 2 \\ \mathcal{H}_1 & \mathcal{H}_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0\ldots 0 & 2\ldots 2 \\ \mathcal{H}_2 & \mathcal{H}_2 \end{pmatrix}$$

are parity-check matrices of quaternary linear perfect codes $\mathcal{D}_1$ and $\mathcal{D}_2$ of dual type $(0, 2\beta; \gamma_1+1, \delta_1)$ and $(0, 2\beta; \gamma_2+1, \delta_2)$, respectively. Since the intersection code $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(0, \beta; i, j)$, the intersection code $\mathcal{D}_1 \cap \mathcal{D}_2$ is of dual type $(0, 2\beta; i+1, j)$. Moreover, taking the permutation $\pi = (1, \beta+1)$, the intersection code $\mathcal{D}_1 \cap \pi(\mathcal{D}_2)$ is of dual type $(0, 2\beta; i+2, j)$, because we are adding the row $\pi(0, \ldots, 0, 2, \ldots, 2)$ of order two in the parity-check matrix of this intersection. $\square$

Lemma 4 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be quaternary linear perfect codes of dual type $(0, \beta; \gamma_1, \delta_1)$ and $(0, \beta; \gamma_2, \delta_2)$ respectively, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(0, \beta; i, j)$, $\beta \geq 4$. Then, there exist two quaternary linear perfect codes of dual type $(0, 4\beta; \gamma_1, \delta_1+1)$ and $(0, 4\beta; \gamma_2, \delta_2+1)$ with intersection codes of dual type $(0, 4\beta; i', j')$ for $(i', j') \in \{(i, j+1), (i, j+2), (i+1, j+1)\}$.*

Proof: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be parity-check matrices of the quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. The matrices

$$\begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_1 & \mathcal{H}_1 & \mathcal{H}_1 \\ 0\ldots 0 & 1\ldots 1 & 2\ldots 2 & 3\ldots 3 \end{pmatrix} \text{ and } \begin{pmatrix} \mathcal{H}_2 & \mathcal{H}_2 & \mathcal{H}_2 & \mathcal{H}_2 \\ 0\ldots 0 & 1\ldots 1 & 2\ldots 2 & 3\ldots 3 \end{pmatrix}$$

are parity-check matrices of quaternary linear perfect codes $\mathcal{D}_1$ and $\mathcal{D}_2$ of dual type $(0, 4\beta; \gamma_1, \delta_1+1)$ and $(0, 4\beta; \gamma_2, \delta_2+1)$, respectively. Since the code $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(0, \beta; i, j)$, the intersection code $\mathcal{D}_1 \cap \mathcal{D}_2$ is of dual type $(0, 4\beta; i, j+1)$. Taking for example the permutation $\pi = (1, \beta+1)$, the intersection code $\mathcal{D}_1 \cap \pi(\mathcal{D}_2)$ is of dual type $(0, 4\beta; i, j+2)$, because we are adding two new independent rows of order four, $v = (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3)$ and $\pi(v)$, to the parity-check matrix of the intersection. Finally, taking the permutation $\sigma = (1, 2\beta+1)$, the intersection code $\mathcal{D}_1 \cap \sigma(\mathcal{D}_2)$ is of dual type $(0, 4\beta; i+1, j+1)$, because the rows, v and $\sigma(v)$, in the parity-check matrix are equivalent to the rows v and $(2, 0, \ldots, 0, 2, 0, \ldots, 0)$ of order four and two, respectively. $\square$

Lemma 5 *For all $m \geq 2$ there exist two quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0, 2^{2m}; 0, m+1)$ and $(0, 2^{2m}; 2m, 1)$ respectively, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a quaternary linear code of dual type $(0, 2^{2m}; \gamma, m+1)$, where γ is any value from m to $2m$ (so any value given by Theorem 6).*

Proof: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be parity-check matrices of the quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual types $(0, 2^{2m}; 0, m+1)$ and $(0, 2^{2m}; 2m, 1)$, respectively. Assume

$$\mathcal{H}_1 = \begin{pmatrix} \overline{1 \dots 1} \\ q_1 \\ \vdots \\ q_m \end{pmatrix} \quad \text{and} \quad \mathcal{H}_2 = \left(\frac{2H_{2m}}{1 \dots 1} \right),$$

where the rows $q_1, \dots, q_m$ form a submatrix that has as columns all the vectors of $\mathbb{Z}_4^m$ ordered lexicographically, and H_{2m} is the parity-check matrix of an extended binary Hamming code of length 2^{2m} whose columns are also ordered lexicographically.

The code $\mathcal{C}_1 \cap \pi(\mathcal{C}_2)$, which has parity-check matrix $\mathcal{H}_1 \parallel \pi(\mathcal{H}_2)$, is of dual type $(0, 2^{2m}; \gamma_\pi, m+1)$, where $\gamma_\pi \in \{m, \dots, 2m\}$, for any permutation π on the set of coordinates.

The rows $2q_1, \dots, 2q_m$ of order two are included in the matrix $\mathcal{H}_2$. So, in the matrix $\mathcal{H}_1 \parallel \mathcal{H}_2$ there are only m independent rows of order two, which means that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a quaternary linear code of dual type $(0, 2^{2m}; m, m+1)$.

For each $i \in \{1, \dots, m\}$, there exists a transposition σ_i that fixes all the rows $2q_j$ for $j = 1, \dots, m$, $j \neq i$, and switches two coordinates that contain different elements in $2q_i$. Taking the permutation $\pi_i = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_i$, the intersection code $\mathcal{C}_1 \cap \pi_i(\mathcal{C}_2)$ is of dual type $(0, 2^{2m}; m+i, m+1)$. $\square$

Lemma 6 *For all $m \geq 2$ and any γ_1, δ_1, such that $\gamma_1 + 2\delta_1 = 2m+2$ and $\delta_1 \geq 1$, there exist two quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0, 2^{2m}; \gamma_1, \delta_1)$ and $(0, 2^{2m}; 0, m+1)$ respectively, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a quaternary linear code of dual type $(0, 2^{2m}; \gamma, \delta)$, where γ and δ are any values*

$$m+1 \leq \delta \leq \delta_1 + m \quad \textit{and}$$

$$max(0, \gamma_1 + \delta_1 - \delta) \leq \gamma \leq \gamma_1 + \delta_1 - \delta + m$$

(so any values given by Theorem 6).

Proof: By Theorem 2, we know that for each $\delta_1 \in \{1, \dots, m+1\}$ there exists a non-isomorphic quaternary linear perfect code of dual type $(0, 2^{2m}; 2(m-\delta_1+1), \delta_1)$. We will prove the statement, by induction on m and for each possible δ_1.

First, for $m = 2$ we need to show the existence of two perfect codes of dual type $(0, 16; 6-2\delta_1, \delta_1)$ and $(0, 16; 0, 3)$ with all possible intersections, for each

$\delta_1 \in \{1,2,3\}$. For $\delta_1 = 1$, we have the result by Lemma 5. For $\delta_1 = 2$, we use Lemma 4 and the codes constructed in Example 3 of dual types $(0,4;2,1)$ and $(0,4;0,2)$ with intersection of dual type $(0,4;1,2)$. Hence, there exist codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0,16;2,2)$ and $(0,16;0,3)$ with intersection codes of dual types $(0,16;1,3)$, $(0,16;1,4)$ and $(0,16;2,3)$. Moreover, taking the quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length 16 with parity-check matrices

$$\mathcal{H}_1 = \left(\begin{array}{cccccccccccccccc} 0&0&0&0&0&0&0&0&2&2&2&2&2&2&2&2 \\ 0&0&0&0&2&2&2&2&0&0&0&0&2&2&2&2 \\ \hline 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3 \end{array} \right)$$

and $\mathcal{H}_2$ below, respectively, and the permutations $\pi_1 = (2,4)(3,5)$, $\pi_2 = (1,2)(3,4)(5,8,7,6)$ and $\pi_3 = (1,2)(3,5)$; the intersection codes $\pi_i(\mathcal{C}_1) \cap \mathcal{C}_2$, $i = 1,2,3$, are of dual type $(0,16;3,3)$, $(0,16;0,4)$ and $(0,16;2,4)$, respectively. This gives all possible values for $m = 2$ and $\delta_1 = 2$. Finally, we prove the result for $m = 2$ and $\delta_1 = 3$. Again by Example 3, there exist (isomorphic) perfect codes of dual type $(0,4;0,2)$ with intersection codes of dual type $(0,4;0,2)$ and $(0,4;0,3)$. Then, by Lemma 4, there exist codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0,16;0,3)$ with intersection codes of dual type $(0,16;i,j)$, for all $0 \leq i \leq 5-j$ and $3 \leq j \leq 5$, except for the case when $i = 2$ and $j = 3$. Moreover, taking the quaternary linear perfect code $\mathcal{C}_2$ of length 16 with parity-check matrix

$$\mathcal{H}_2 = \left(\begin{array}{cccccccccccccccc} \hline 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3 \\ 0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3 \end{array} \right)$$

and the permutation $\pi = (7,13,15)(8,14,16)$, the intersection code $\mathcal{C}_2 \cap \pi(\mathcal{C}_2)$ is of dual type $(0,16;2,3)$. So, the result is true for $m = 2$.

Now, we assume that the result is true for perfect codes of dual type $(0,2^{2(m-1)};\gamma_1,\delta_1)$ and $(0,2^{2(m-1)};0,m)$, for each $\delta_1 \in \{1,\ldots,m\}$ and $\gamma_1 = 2(m-\delta_1)$. Let there exist intersection codes of dual type $(0,2^{2(m-1)};i',j')$, for all

$$m \leq j' \leq \delta_1 + m - 1 \quad \text{and}$$

$$max(0, \gamma_1 + \delta_1 - j') \leq i' \leq \gamma_1 + \delta_1 - j' + m - 1.$$

Then, by Lemma 4, there exist codes $\mathcal{D}_1$ and $\mathcal{D}_2$ of dual type $(0,2^{2m};\gamma_1,\delta_1+1)$ and $(0,2^{2m};0,m+1)$ with intersection codes of dual type $(0,2^{2m};i,j)$, for all

$$m+1 \leq j \leq (\delta_1+1)+m \quad \text{and}$$

$$max(0, \gamma_1 + (\delta_1 + 1) - j) \leq i \leq \gamma_1 + (\delta_1 + 1) - j + m,$$

given any $\delta_1 + 1 \in \{2, \ldots, m+1\}$. If $\delta_1 = 1$ then $\gamma_1 = 2m$ (see Theorem 2) and by Lemma 5, we have the result for two perfect codes of dual type $(0, 2^{2m}; \gamma_1, 1)$ and $(0, 2^{2m}; 0, m+1)$. So, the result is true for any $\delta_1 \in \{1, \ldots, m+1\}$. $\square$

Theorem 7 *For all $t \geq 4$ and any $\gamma_1, \delta_1, \gamma_2, \delta_2$, such that $\gamma_1 + 2\delta_1 = \gamma_2 + 2\delta_2 = t + 1$ and $\delta_1, \delta_2 \geq 1$, there exist two quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(0, 2^{t-1}; \gamma_1, \delta_1)$ and $(0, 2^{t-1}, \gamma_2, \delta_2)$ respectively, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a quaternary linear code of dual type $(0, 2^{t-1}; \gamma, \delta)$, where*

$$\begin{aligned} \delta \in \{max(\delta_1, \delta_2), \ldots, \delta_1 + \delta_2 - 1\} \quad \textit{and} \\ max\big(\delta, max(\gamma_1 + \delta_1, \gamma_2 + \delta_2)\big) \leq \gamma + \delta \leq \gamma_1 + \gamma_2 + \delta_1 + \delta_2 - 1. \end{aligned} \tag{17}$$

Proof: By Theorem 2, we know that for each $t \geq 4$ there are $\lfloor (t+1)/2 \rfloor$ non-isomorphic quaternary linear perfect codes of length $\beta = 2^{t-1}$. Specifically, for each $\bar{\delta} \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$ there exists one code of dual type $(0, 2^{t-1}; t + 1 - 2\bar{\delta}, \bar{\delta})$.

We will prove the statement by induction on $t \geq 4$, and we need to show the result is true for the initial cases $t = 4$ and $t = 5$.

For $t = 4$, we have two non-isomorphic quaternary linear perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ given by $\bar{\delta} = 1$ and $\bar{\delta} = 2$, respectively, and with parity-check matrices

$$\mathcal{H}_1 = \left(\frac{2H_3}{1 \ldots 1} \right) \quad \text{and} \quad \mathcal{H}_2 = \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{array} \right),$$

where H_3 is a parity-check matrix of an extended binary Hamming code of length 8. When $\delta_1 = \delta_2 = 1$, by using the proof of Theorem 5, we have intersections of dual type $(0, 8; r, 1)$ for any r from $\{3, 4, 5, 6\}$. When $\delta_1 = \delta_2 = 2$, using the quaternary linear perfect code of dual type $(0, 4; 0, 2)$, the intersection codes in Example 3 and Lemma 3, we get all the possible intersection codes for this case, except the intersections of dual types $(0, 8; 3, 2)$ and $(0, 8; 0, 3)$. However, taking the permutations $\pi = (1, 5)(2, 4)$ and $\sigma = (1, 2)(3, 4)$, the codes $\mathcal{C}_2 \cap \pi(\mathcal{C}_2)$ and $\mathcal{C}_2 \cap \sigma(\mathcal{C}_2)$ are of dual types $(0, 8; 3, 2)$ and $(0, 8; 0, 3)$, respectively. Finally, when $\delta_1 = 1$ and $\delta_2 = 2$, using the perfect codes of dual types $(0, 4; 2, 1)$ and $(0, 4; 0, 2)$, the intersection code of dual type $(0, 4; 1, 2)$ in Example 3 and Lemma 3, there exist all possible intersections codes for this case, except the intersection code of dual type $(0, 8; 4, 2)$. Taking the permutation $\tau = (1, 2)(4, 5)$, the code $\tau(\mathcal{C}_1) \cap \mathcal{C}_2$ is of this missing type.

Similarly, for $t = 5$ we can find, by direct search, all the possible intersection codes fulfilling the statement. We avoid to write here the complete list.

Now, we assume the result is true for quaternary linear perfect codes of length $\beta = 2^{t-2}$ and $\beta = 2^{t-3}$ ($t > 5$). So, this means we are assuming that for any $\delta_1, \delta_2 \in \{1, \ldots, \lfloor t/2 \rfloor\}$ we have quaternary linear perfect codes of dual type $(0, 2^{t-2}; t - 2\delta_1, \delta_1)$ and $(0, 2^{t-2}; t - 2\delta_2, \delta_2)$ with intersection codes of dual type $(0, 2^{t-2}; i', j')$, for all i', j' fulfilling equations (17). By Lemma 3, we have quaternary linear perfect codes of dual type $(0, 2^{t-1}; t + 1 - 2\delta_1, \delta_1)$ and $(0, 2^{t-1}; t + 1 - 2\delta_2, \delta_2)$ with intersection codes of dual type $(0, 2^{t-1}; i, j)$, for all $j = j'$ and $i \in \{i' + 1, i' + 2\}$, where

$$max(\delta_1, \delta_2) \leq j \leq \delta_1 + \delta_2 - 1 \quad \text{and}$$

$$max(1, max(t + 1 - \delta_1, t + 1 - \delta_2) - j) \leq i \leq 2(t + 1) - \delta_1 - \delta_2 - j - 1.$$

So, we obtain all the possible types for the intersection, except for $i = 0$ and $j \geq max(t + 1 - \delta_1, t + 1 - \delta_2)$. But, in this exceptional case, we have $j \geq max(\delta_1, \delta_2) + 1$ because, otherwise, assuming $j < \delta_1 + 1$ we would have $\delta_1 + 1 > t + 1 - \delta_1$ and $2\delta_1 > t$ which is a contradiction.

Also, from induction hypothesis, we can assume that for any $\delta_1, \delta_2 \in \{2, \ldots, \lfloor (t + 1)/2 \rfloor\}$ we have quaternary linear perfect codes of dual type $(0, 2^{t-3}; (t - 1) - 2(\delta_1 - 1), \delta_1 - 1)$ and $(0, 2^{t-3}; (t - 1) - 2(\delta_2 - 1), \delta_2 - 1)$ with intersection codes of dual type $(0, 2^{t-3}; i', j')$, for all i', j' fulfilling equations (17). By Lemma 4, we have quaternary linear perfect codes of dual type $(0, 2^{t-1}; t + 1 - 2\delta_1, \delta_1)$ and $(0, 2^{t-1}; t + 1 - 2\delta_2, \delta_2)$ with intersection codes of dual type $(0, 2^{t-1}; i, j)$, for all $(i, j) \in \{(i', j' + 1), (i', j' + 2), (i' + 1, j' + 1)\}$, where

$$max(\delta_1, \delta_2) + 1 \leq j \leq \delta_1 + \delta_2 - 1 \quad \text{and}$$

$$max(0, max(t + 1 - \delta_1, t + 1 - \delta_2) - j) \leq i \leq 2(t + 1) - \delta_1 - \delta_2 - j - 1.$$

Notice that when t is even then $\lfloor t/2 \rfloor$ coincides with $\lfloor (t + 1)/2 \rfloor$ so, in this case the proof is finished. When t is odd we need to prove the statement for $\delta_1 \in \{1, \ldots, (t + 1)/2\}$ and $\delta_2 = (t + 1)/2$, which is straightforward from Lemma 6. $\square$

4 Intersection of additive extended perfect codes with $\alpha \neq 0$

In this section we consider the intersection problem for additive extended perfect codes such that $\alpha \neq 0$. We also investigate the abelian group struc-

ture for the intersection of such codes. Again, all statements presented in Section 2 are valid for this case with some small changes.

As we said before, the extended perfect $\mathbb{Z}_4$-linear codes and the extended codes of the perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes could be seen as additive extended perfect codes after the Gray map. In the first case, the quaternary all-ones vector belongs to the code and also to the quaternary dual code. In the second case, the vector with binary ones in the binary part and quaternary twos in the quaternary part is always in the code and also in the additive dual code.

Given an additive extended perfect code $\mathcal{C}$ of dual type $(\alpha, \beta; \gamma, \delta)$ with $\alpha \neq 0$, we always have $\alpha+2\beta = 2^t$; $\gamma+2\delta = t+1$; $\alpha = 2^r$ and $\gamma+\delta = r+1$ (see Theorem 1). Hence, given the parameters α and β, all the additive extended perfect codes with these parameters must have the same parameters γ and δ.

Proposition 2 is transformed in the next proposition:

Proposition 4 *For any two additive extended perfect codes such that their additive dual codes are $\mathcal{C}_1$ and $\mathcal{C}_2$ of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta})$ with $\alpha \neq 0$, the code $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is an additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, where*

$$\begin{cases} \textit{if } \bar{\delta} = 0 & \textit{then} \quad \delta = 0 \textit{ and } \bar{\gamma} \leq \kappa = \gamma \leq 2\bar{\gamma} - 1, \\ \textit{if } \bar{\delta} = 1 & \textit{then} \quad \delta = 1 \textit{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma}, \\ \textit{if } \bar{\delta} > 1 & \textit{then} \quad \delta \in \{\bar{\delta}, \dots, 2\bar{\delta}\} \textit{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma} + 2\bar{\delta} - \delta - 1. \end{cases}$$

Proof: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are the additive dual codes of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta})$ with $\alpha \neq 0$, then there exist values $2 \leq r \leq t \leq 2r$ (see Theorem 1) such that $\alpha = 2^r$, $\beta = 2^{t-1} - 2^{r-1}$, $\bar{\gamma} = 2r - t + 1$ and $\bar{\delta} = t - r$.

For these codes Proposition 2 becomes

$$\delta \in \{\bar{\delta}, \dots, min(2\bar{\delta}, \beta)\}, \tag{18}$$

$$max(\delta, \bar{\gamma} + \bar{\delta}) \leq \kappa + \delta \leq min(2(\bar{\gamma} + \bar{\delta}) - 1, \alpha + \beta) \tag{19}$$

$$\text{and} \quad \kappa + \delta \leq \gamma + \delta \leq min(2(\bar{\gamma} + \bar{\delta}) - 1, \alpha + \beta), \tag{20}$$

because the vector with binary ones in the binary part and quaternary twos in the quaternary part is always in $\mathcal{C}_1$ and $\mathcal{C}_2$.

The lower bound in equation (19) can be improved. The $\bar{\gamma}$ vectors of order two in $\mathcal{C}_1$ or $\mathcal{C}_2$ are necessarily independent of the $\bar{\delta}$ vectors of order four on the other code, respectively, so the lower bound becomes $\bar{\gamma} + \delta$. Moreover, for $t \geq 3$ we have $\alpha+\beta = 2^{t-1}+2^{r-1} = 2^{r-1}(2^{t-r}+1) \geq 2r+1 = 2(\bar{\gamma}+\bar{\delta})-1$, so equations (19) and (20) became

$$\bar{\gamma} + \delta \leq \kappa + \delta \leq \gamma + \delta \leq 2(\bar{\gamma} + \bar{\delta}) - 1. \tag{21}$$

If $\bar{\delta}=0$ from equation (18) we have $\delta=0$ and $\beta=0$. From equation (21) we can write $\bar{\gamma}\leq\kappa\leq\gamma\leq 2\bar{\gamma}-1$. In this case, we can add that $\kappa=\gamma$, since the $\gamma-\kappa$ vectors of order two are the ones that are independent when we restrict them to the quaternary part, but there is not quaternary part because $\beta=0$.

If $\bar{\delta}=1$ it is not possible to have two independent vectors of order four in $\langle\mathcal{C}_1,\mathcal{C}_2\rangle$, so $\delta=1$ and from equation (21) we have $\bar{\gamma}\leq\kappa\leq\gamma\leq 2\bar{\gamma}$.

If $\bar{\delta}>1$ then $2\bar{\delta}=2(t-r)\leq 2^{t-1}-2^{r-1}=\beta$. So, from equation (18) we obtain $\delta\in\{\bar{\delta},\ldots,2\bar{\delta}\}$ and from equation (21) we obtain $\bar{\gamma}\leq\kappa\leq\gamma\leq 2\bar{\gamma}+2\bar{\delta}-\delta-1$. $\square$

Recall that the parameters of an additive code can be computed from the parameters of its additive dual code using equations (5), so we can establish the following theorem.

Theorem 8 *For any two additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha,\beta;\bar{\gamma},\bar{\delta})$ with $\alpha\neq 0$, the intersection code $\mathcal{C}_1\cap\mathcal{C}_2$ is of dual type $(\alpha,\beta;\gamma,\delta;\kappa)$, where γ,δ,κ satisfy the bounds given by Proposition 4.*

Example 4 *For $t=3$ there are two isomorphic additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ given by $\bar{\delta}=0$ and $\bar{\delta}=1$, so of dual types $(8,0;4,0)$ and $(4,2;2,1)$, respectively. The code $\mathcal{C}_1$ corresponds to an extended binary Hamming code of length 8, so we have intersections codes of dual type $(8,0;\gamma,0;\kappa)$ for any value $\gamma=\kappa$ from 4 to 7 (see [EV98] or (1)). By an exhaustive search and taking $\mathcal{H}_2$ as a parity-check matrix of $\mathcal{C}_2$, the possible intersection codes $\mathcal{C}_2\cap\pi(\mathcal{C}_2)$, which have parity-check matrices $\mathcal{H}_2\parallel\pi(\mathcal{H}_2)$, are of dual type $(4,2;\gamma,\delta;\kappa)$, where*

$$\mathcal{H}_2=\left(\begin{array}{cccc|cc}1&1&1&1&2&2\\0&0&1&1&0&2\\ \hline 0&1&0&1&1&1\end{array}\right)$$

and

γ	δ	κ	π
2	1	2	Id
3	1	2	$(1,2)$
3	1	3	$(1,3)$
4	1	3	$(1,2,3)$.

Lemma 7 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive extended perfect codes of dual type $(\alpha,\beta;\gamma,\delta)$ with $\alpha\neq 0$, $\alpha+2\beta=2^t$ and $t\geq 3$, such that $\mathcal{C}_1\cap\mathcal{C}_2$ is of dual type $(\alpha,\beta;i,j;k)$. Then, there exist two additive extended perfect codes of dual type $(2\alpha,2\beta;\gamma+1,\delta)$ with intersection codes of dual type $(2\alpha,2\beta;i',j';k')$ for $(i',j',k')\in\{(i+1,j,k+1),(i+2,j;k+1),(i+2,j;k+2)\}$.*

Proof: Let $\mathcal{H}_1=(\mathcal{H}_{1,\alpha}|\mathcal{H}_{1,\beta})$ and $\mathcal{H}_2=(\mathcal{H}_{2,\alpha}|\mathcal{H}_{2,\beta})$ be parity-check matrices of the additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. The

matrices

$$\left(\begin{array}{cc|cc} 0\dots0 & 1\dots1 & 0\dots0 & 2\dots2 \\ \mathcal{H}_{1,\alpha} & \mathcal{H}_{1,\alpha} & \mathcal{H}_{1,\beta} & \mathcal{H}_{1,\beta} \end{array}\right) \text{ and } \left(\begin{array}{cc|cc} 0\dots0 & 1\dots1 & 0\dots0 & 2\dots2 \\ \mathcal{H}_{2,\alpha} & \mathcal{H}_{2,\alpha} & \mathcal{H}_{2,\beta} & \mathcal{H}_{2,\beta} \end{array}\right)$$

are parity-check matrices of additive extended perfect codes $\mathcal{D}_1$ and $\mathcal{D}_2$ of dual type $(2\alpha, 2\beta; \gamma+1, \delta)$. Using similar arguments than in Lemma 3, we can take the permutations $\pi = Id$, $\pi = (1, \alpha+1)$ and $\pi = (2\alpha+1, 2\alpha+\beta+1)$, in order to obtain the intersection codes $\mathcal{D}_1 \cap \pi(\mathcal{D}_2)$ of dual type $(2\alpha, 2\beta; i+1, j; k+1)$, $(2\alpha, 2\beta; i+2, j; k+2)$ and $(2\alpha, 2\beta; i+2, j; k+1)$, respectively. $\square$

Lemma 8 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive extended perfect codes of dual type $(\alpha, \beta; 1, \delta)$ with $\alpha \neq 0$, $\alpha + 2\beta = 2^t$ and $t \geq 3$, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(\alpha, \beta; i, j; k)$. Then, there exist two additive extended perfect codes of dual type $(2\alpha, \alpha+4\beta; 1, \delta+1)$ with intersection codes of dual type $(2\alpha, \alpha+4\beta; i', j'; k')$ for $(i', j', k') \in \{(i, j+1; k), (i, j+2; k), (i+1, j+1; k), (i+1, j+1; k+1)\}$.*

Proof: Let $\mathcal{H}_1 = (\mathcal{H}_{1,\alpha}|\mathcal{H}_{1,\beta})$ and $\mathcal{H}_2 = (\mathcal{H}_{2,\alpha}|\mathcal{H}_{2,\beta})$ be parity-check matrices of the additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, such that they contain the vector $(1\dots1|2\dots2)$ in the first row. The matrices

$$\left(\begin{array}{cc|ccccc} \mathcal{H}_{1,\alpha} & \mathcal{H}_{1,\alpha} & 2\mathcal{H}_{1,\alpha} & \mathcal{H}_{1,\beta} & \mathcal{H}_{1,\beta} & \mathcal{H}_{1,\beta} & \mathcal{H}_{1,\beta} \\ 0\dots0 & 1\dots1 & 1\dots1 & 0\dots0 & 1\dots1 & 2\dots2 & 3\dots3 \end{array}\right) \text{ and}$$

$$\left(\begin{array}{cc|ccccc} \mathcal{H}_{2,\alpha} & \mathcal{H}_{2,\alpha} & 2\mathcal{H}_{2,\alpha} & \mathcal{H}_{2,\beta} & \mathcal{H}_{2,\beta} & \mathcal{H}_{2,\beta} & \mathcal{H}_{2,\beta} \\ 0\dots0 & 1\dots1 & 1\dots1 & 0\dots0 & 1\dots1 & 2\dots2 & 3\dots3 \end{array}\right)$$

are parity-check matrices of additive extended perfect codes $\mathcal{D}_1$ and $\mathcal{D}_2$ of dual type $(2\alpha, \alpha+4\beta; 1, \delta+1)$. Notice that the first row in these two matrices is again the vector $(1\dots1|2\dots2)$. The codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are of binary length $\alpha + 2\beta = 2^t$, so the codes $\mathcal{D}_1$ and $\mathcal{D}_2$ are of binary length $2\alpha + 2(\alpha+4\beta) = 4(\alpha+2\beta) = 2^{t+2}$.

Using similar arguments than in Lemma 4, we can take the permutations $\pi = Id$, $\pi = (2\alpha+1, 3\alpha+\beta+1)$, $\pi = (3\alpha+1, 3\alpha+2\beta+1)$ and $\pi = (1, \alpha+1)$, in order to obtain the intersection codes $\mathcal{D}_1 \cap \pi(\mathcal{D}_2)$ of dual type $(2\alpha, \alpha+4\beta; i, j+1; k)$, $(2\alpha, \alpha+4\beta; i, j+2; k)$, $(2\alpha, \alpha+4\beta; i+1, j+1; k)$ and $(2\alpha, \alpha+4\beta; i+1, j+1; k+1)$, respectively. $\square$

Lemma 9 *For all $m \geq 2$ there exist two additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(2^m, 2^{2m-1} - 2^{m-1}; 1, m)$, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is an additive extended perfect code of dual type $(2^m, 2^{2m-1} - 2^{m-1}; \gamma, \delta; \kappa)$, where γ, δ, κ are any values*

$$\delta \in \{m, \ldots, 2m\} \quad \textit{and} \quad 1 \leq \kappa \leq \gamma \leq 2m - \delta + 1$$

(so any values given by Theorem 8).

Proof: By Lemma 8 and the same argument than in Lemma 6, the result follows. We only need to prove it for $m = 2$. Let $\mathcal{C}_1$ be the perfect code of dual type $(4, 6; 1, 2)$ with parity-check matrix

$$\mathcal{H}_1 = \left(\begin{array}{cccc|cccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 2 & 3 & 1 & 1 \end{array} \right).$$

The possible intersection codes $\mathcal{C}_1 \cap \pi(\mathcal{C}_1)$ are of dual type $(4, 6; \gamma, \delta; \kappa)$, where

γ	δ	κ	π
1	2	1	Id
2	2	1	$(5,7)$
2	2	2	$(1,2)$
3	2	1	$(1,3)(6,9)(8,10)$
3	2	2	$(1,3)(5,7)$
3	2	3	$(1,2,3)$
1	3	1	$(5,6)$
2	3	1	$(5,6)(9,10)$
2	3	2	$(1,2)(6,9)$
1	4	1	$(5,6)(7,9)$.

□

Theorem 9 *For all $t \geq 4$ and any $\bar{\gamma}, \bar{\delta}$, such that $\bar{\gamma} + 2\bar{\delta} = t + 1$, there exist two additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \bar{\gamma}, \bar{\delta})$, with $\alpha \neq 0$ and $\alpha + 2\beta = 2^t$, such that $\mathcal{C}_1 \cap \mathcal{C}_2$ is an additive code of dual type $(\alpha, \beta; \gamma, \delta; \kappa)$, where*

$$\begin{cases} \textit{if } \bar{\delta} = 0 & \textit{then} \quad \delta = 0 \textit{ and } \bar{\gamma} \leq \kappa = \gamma \leq 2\bar{\gamma} - 1, \\ \textit{if } \bar{\delta} = 1 & \textit{then} \quad \delta = 1 \textit{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma}, \\ \textit{if } \bar{\delta} > 1 & \textit{then} \quad \delta \in \{\bar{\delta}, \ldots, 2\bar{\delta}\} \textit{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma} + 2\bar{\delta} - \delta - 1. \end{cases}$$

except for codes of dual type $(8, 4; 3, 1)$ for which the intersection code of dual type $(8, 4; 6, 1; 3)$ does not exist.

Proof: By Theorem 1, we know that for each $t \geq 4$ there are $\lfloor (t+2)/2 \rfloor$ non-isomorphic additive extended perfect codes of binary length $\alpha + 2\beta = 2^t$. Specifically, for each $\bar{\delta} \in \{0, \ldots, \lfloor t/2 \rfloor\}$ there exists one of dual type $(2^{t-\bar{\delta}}, 2^{t-1} - 2^{t-\bar{\delta}-1}; t+1-2\bar{\delta}, \bar{\delta})$. Notice that when $\alpha + 2\beta = 2^{2m-1}$ we have $\bar{\delta} \in \{0, \ldots, m-1\}$ and when $\alpha + 2\beta = 2^{2m}$ we have $\bar{\delta} \in \{0, \ldots, m\}$.

For $t = 4$, we have three non-isomorphic additive extended perfect codes given by $\bar{\delta} = 0, 1$ and 2. For $\bar{\delta} = 0$, the code corresponds to an extended binary Hamming code and for these codes the result was proved in [EV98] (see (1)). For $\bar{\delta} = 1$, using Lemma 7 and the codes constructed in Example 4, we can obtain intersection codes of all different dual types except for $(8, 4; 6, 1; 3)$ and $(8, 4; 6, 1; 6)$. By an exhaustive search, the intersection code of dual type $(8, 4; 6, 1; 3)$ does not exist. However, taking the permutation $\pi = (1, 8, 7, 6, 5, 4, 3)$ the intersection code $\mathcal{C}_1 \cap \pi(\mathcal{C}_1)$ is of dual type $(8, 4; 6, 1; 6)$, where $\mathcal{C}_1$ is the perfect code of dual type $(8, 4; 3, 1)$ constructed using Lemma 7.

Like in Theorem 7, in order to use induction in this proof, since for $t = 4$ and $\bar{\delta} = 1$ the intersection code of dual type $(8, 4; 6, 1; 3)$ does not exist, we need to show the existence of the intersection code of dual type $(16, 8; 8, 1; 4)$ for $t = 5$ and $\bar{\delta} = 1$. Again taking the permutation $\pi = (1, 13, 10, 5)(2, 14, 9, 6)(3, 16, 12, 8)$
$(4, 15, 11, 7)(17, 22, 18, 20, 24, 21)(19, 23)$ the intersection code $\mathcal{C}_2 \cap \pi(\mathcal{C}_2)$ is of dual type $(16, 8; 8, 1; 4)$, where $\mathcal{C}_2$ is the perfect code of dual type $(16, 8; 4, 1)$ constructed using Lemma 7. Finally, by Lemmas 7 and 9 and using similar arguments than in Theorem 7, the result follows. $\square$

Next theorem describes the intersection numbers for the additive extended perfect codes with $\alpha \neq 0$.

Theorem 10 *For any $t \geq 3$ and any additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \gamma, \delta)$, with $\alpha \neq 0$ and $\alpha + 2\beta = 2^t$, it is true that*

$$\begin{cases} \textit{if } \delta = 1 & \textit{then} \quad 2^{\alpha+2\beta-2t} \leq \eta(\mathcal{C}_1, \mathcal{C}_2) \leq 2^{\alpha+2\beta-t-1}, \\ \textit{if } \delta \neq 1 & \textit{then} \quad 2^{\alpha+2\beta-2t-1} \leq \eta(\mathcal{C}_1, \mathcal{C}_2) \leq 2^{\alpha+2\beta-t-1}. \end{cases}$$

Proof: It is straightforward from Proposition 4 and using the same argument than in Theorem 3. $\square$

Next result is to point out that the bounds in the previous theorem are tight. Moreover, we show that there exist such codes for any possible intersection number between these bounds. It is easy and straightforward to settle this from Theorem 9.

Theorem 11 *For any $t \geq 3$ there exist two additive extended perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \gamma, \delta)$, with $\alpha \neq 0$ and $\alpha + 2\beta = 2^t$, such that $\eta(\mathcal{C}_1, \mathcal{C}_2) = 2^{\alpha+2\beta-l}$, where l is any value such that*

$$\begin{cases} \text{if } \delta = 1 & \text{then} \quad l \in \{t+1, \dots, 2t\}, \\ \text{if } \delta \neq 1 & \text{then} \quad l \in \{t+1, \dots, 2t+1\}. \end{cases}$$

Finally, to end this section, notice that the usual binary perfect codes of length $2^t - 1$ are obtained from the extended ones by puncturing one coordinate. The additive perfect codes can also be constructed taking the parity-check matrix of an additive extended perfect code, deleting the row with ones in the binary part and twos in the quaternary part and also deleting one column in the binary part.

Given any two additive perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \bar{\gamma}, \bar{\delta})$ with $\alpha \neq 0$ and $\alpha + 2\beta = 2^t - 1$, we can construct the extended codes $\mathcal{C}'_1$ and $\mathcal{C}'_2$, respectively. The codes $\mathcal{C}'_1$ and $\mathcal{C}'_2$ are of dual type $(\alpha+1, \beta; \bar{\gamma}+1, \bar{\delta})$.

Using the theorems that we established before for additive extended perfect codes, it is easy to get the same results for the intersection codes $\mathcal{C}_1 \cap \mathcal{C}_2$. We can summarize these results with the following theorem:

Theorem 12 *For any two additive perfect codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of dual type $(\alpha, \beta; \bar{\gamma}, \bar{\delta})$ with $\alpha \neq 0$ and $\alpha + 2\beta = 2^t - 1$, the intersection code $\mathcal{C}_1 \cap \mathcal{C}_2$ is of dual type $(\alpha, \beta; \gamma, \delta; \kappa)$, where γ, δ, κ satisfy:*

$$\begin{cases} \text{if } \bar{\delta} = 1 & \text{then} \quad \delta = 1 \text{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma} + 1, \\ \text{if } \bar{\delta} \neq 1 & \text{then} \quad \delta \in \{\bar{\delta}, \dots, 2\bar{\delta}\} \text{ and } \bar{\gamma} \leq \kappa \leq \gamma \leq 2\bar{\gamma} + 2\bar{\delta} - \delta. \end{cases}$$

For all $t \geq 4$ and any values for γ, δ, κ between these bounds there exist additive perfect codes the intersection of which attains the prescribed values, except for codes of dual type $(7, 4; 2, 1)$ for which the intersection code of dual type $(7, 4; 5, 1; 2)$ does not exist.

This last theorem includes the binary Hamming codes (when $\delta = 0$ and, so, $\beta = 0$), and we can see it as a generalization of the solution of the intersection problem for binary Hamming codes given by Etzion and Vardy (see (1)).

5 Conclusions

In this paper we continue studying the intersection problem for codes initiated in [EV98] (where the authors proposed to find the intersection numbers

for binary perfect codes) and investigated in [BYE97, AHS05, AHS06, PV06, SL06].

Given two additive perfect codes we compute not only the possibilities for the intersection number, but also the abelian group structure of this intersection. We settle the problem for non-extended and extended additive perfect codes, which means that we solved the problem for perfect $\mathbb{Z}_4$-linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

There are still some interesting problems about this topic as, for example, the problem of finding the abelian group structure of the intersection for additive Hadamard codes, so the dual codes of the additive extended perfect codes studied in this paper. Although we know the relationship between the parameters of a given additive code and its additive dual, and it would be easy to find appropriate lower and upper bounds for the intersection structure, it is not straightforward to construct all the codes which have the desired parameters. This last point needs further research and currently we are working on that.

References

[AHS05] S. V. Avgustinovich, O. Heden, F. I. Solov'eva, "On intersections of perfect binary codes", *Bayreuther Mathematische Schriften*, vol. 71, pp. 8–13, 2005.

[AHS06] S. V. Avgustinovich, O. Heden, F. I. Solov'eva, "On intersection problem for perfect binary codes", *Des. Codes Crypt.*, vol. 39, pp. 317–322, 2006.

[BYE97] E. Bar-Yahalom, T. Etzion, "Intersection of isomorphic linear codes", *Journal of Comb. Theory*, Series A 80, pp. 247–256, 1997.

[BF03] J. Borges and C. Fernandez, "Punctured extended 1-perfect $\mathbb{Z}_4$-linear codes", *Technical Report PIRDI-3/02*, UAB, Spain, 2002.

[BF+06] J. Borges, C. Fernández, J. Pujol, J. Rifà and M. Villanueva, "On the duality of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes", *Technical Report PIRDI-1/06*, UAB, Spain, 2006.

[BR99] J. Borges and J. Rifà, "A characterization of 1-additive perfect codes", *IEEE Trans. Information Theory*, vol. 45(5), pp. 1688–1697, 1999.

[Del73] P. Delsarte, "An algebraic approach to the association schemes of coding theory", *Philips Research Rep. Suppl.*, vol. 10, 1973.

[EV98] T. Etzion, A.Vardy, “On perfect codes and tilings: problems and solutions”, *SIAM I. Discrete Math.*, vol. 11, No 2, pp. 205–223, 1998.

[HK+94] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, “The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals and related codes”, *IEEE Trans. Information Theory*, vol. 40, pp. 301–319, 1994.

[Kr01] D. S. Krotov, “$\mathbb{Z}_4$-linear Hadamard and extended perfect codes”, *Proc. of the International Workshop on Coding and Cryptography*, Paris (France), Jan. 8-12, pp. 329–334, 2001.

[PV06] K. T. Phelps and M. Villanueva, “Intersection of Hadamard codes”, submitted to *IEEE Trans. Information Theory*, 2006.

[RP97] J. Rifà and J. Pujol, “Translation invariant propelinear codes”, *IEEE Trans. Information Theory*, vol. 43, pp. 590–598, 1997.

[SL06] F. I. Solov'eva and A. V. Los', “On intersections of q-ary perfect codes”, *Proc. Tenth Int. Workshop “Algebraic and Combinatorial Coding Theory”*. Zvenigorod, Russia. September, pp. 244-247, 2006.